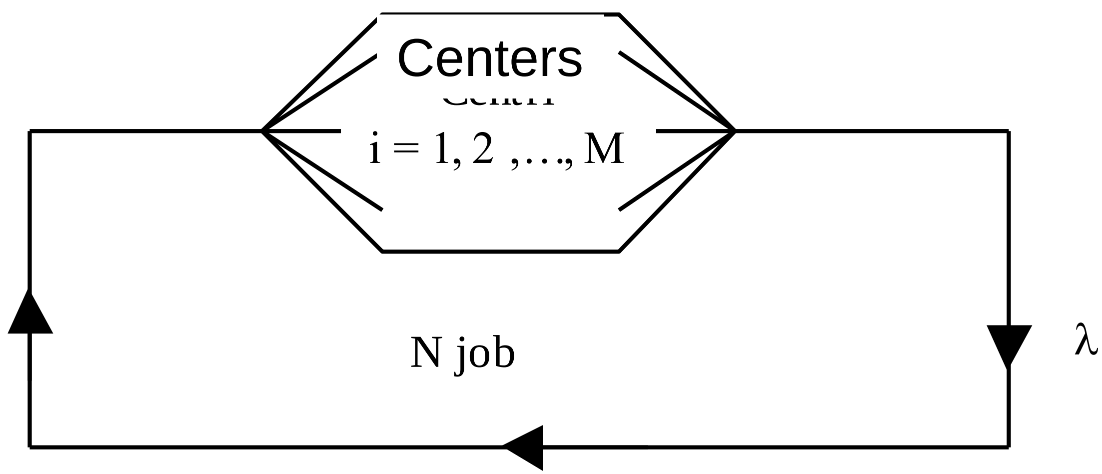
Calculation of Closed QN performance indices

The closed QN performance indices of interest (that we distinguish in *local and global*) are:

- ***throughput***
- ***populations***
- ***waiting times***
- ***response times***

Calculation of performance indices

Let's consider the generic representation of the *closed* QN model



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Notation memo

- Remind we noted the generic state s_i of a QN with M centers by

$$s_i = (n_1, n_2, \ldots., n_M)$$

- And used notation **n** bold to denote vector

$$\mathbf{n} = (n_1, n_2, \ldots., n_M)$$

So in many cases one can find symbol **n** in place of s_i

Local indices

Local indices are the performance indices of a given center *i*.

Starting from the state probability vectors, it is possible to obtain the ***distribution*** $\boldsymbol{\{p_i(n_i)\}}$ ***of the number of users*** $\boldsymbol{n_i}$ ***in the center i***, for i=1, …, M

Local indices

This distribution is obtainable as the marginal distribution of the $\pi(\mathbf{n})$ and written:

$$p_i(k) = \sum_{\forall \mathrm{n} \in E \,|\, n_i = k} \pi(\mathrm{n}) \qquad i = 1, \ldots, M \qquad k = 0, \ldots, N$$

also denoted $\{p_i(n_i),\ n_i=0,\ldots,N\}$ or more precisely $p_i(n_i(N))$ to highlight the fact that the solution is relative to a given value of N (jobs in the system)

Local indices

From the previous relation, it is immediate to derive the ***utilization coefficient*** ρ_i ***of center i***, by means of the relations that were given to calculate the **mean number of** busy processors in a center (less or equal to 1 in a single-processor center and less or equal to m in a m-processors center).

Local indices

It is therefore possible to write the following for the case of a center with m_i processors

$$\rho_i(N)_{multiple} = \frac{1}{m_i} \sum_{n_i=1}^{m_i-1} n_i\, p_i(n_i) + \sum_{n_i=m_i}^{N} p_i(n_i)$$

Local indices

And the following for the case of the center with a single processor (m_i=1)

$$\rho_i(N)_{single} = \sum_{n_i=1}^{N} p_i(n_i) \stackrel{\Delta}{=} p_i(n_i \geq 1)$$

Local indices

From these, it is possible to obtain the ***throughput*** of the center *i*, by applying the law which links the utilization factor ρ to λ and μ.

We will obtain in the case of a single processor center:

$$\lambda_i(N)_{single} = \mu_i \rho_i(N)_{single}$$

Local indices

And in the case of a multiple center

$$\lambda_i(N)_{multiple} = m_i \, \mu_i \, \rho_i(N)_{multiple}$$

With $i=1, \ldots, M$ and with μ_i service rate of each of the m_i processors

Local indices

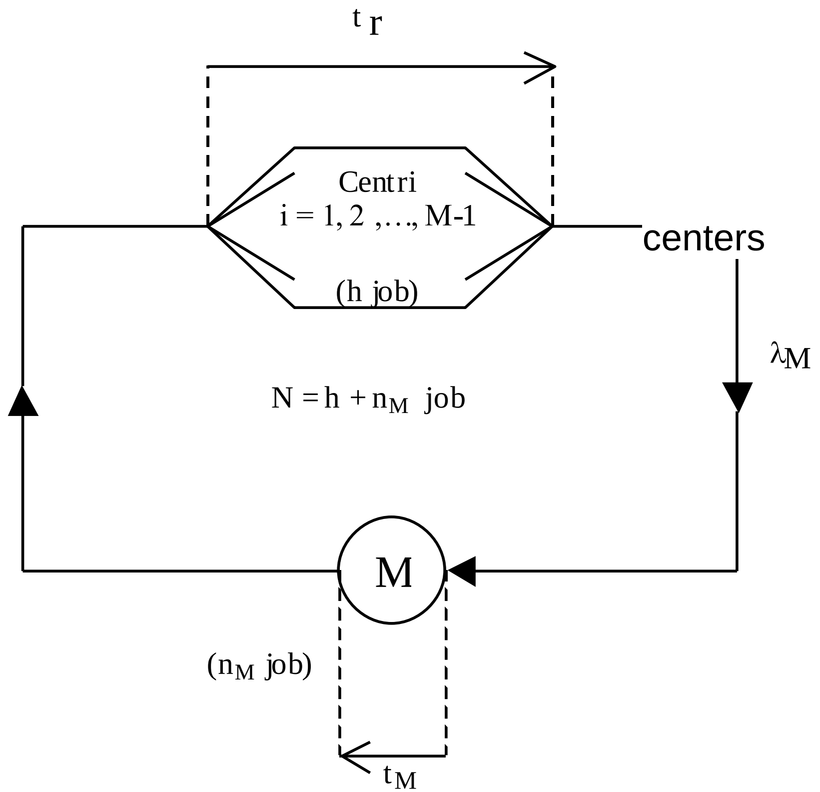
Lastly, we can express the ***mean population of center i*** and then, by applying ***Little's law***, the ***mean response time of the same center***

$$E(n_i(N)) = \sum_{n_i=1}^{N} n_i \; p_i(n_i)$$ ***Mean population of center i***

Mean response time of center i $$E(t_i(N)) = \frac{E(n_i(N))}{\lambda_i(N)}$$

Global indices

For global indices, we intend the performance indices of the entire installation. To this scope we consider the generic *closed* QN model with a reference center M as shown



Global indices

The model in examination highlights a center (center i = M) that will be used as the reference-center for the calculation of the *global indices*

We define ***response time (t_r)*** *of the installation*, ***"seen" by center M*** as the time needed by a job to return to center M after leaving M, in other words the time to cross the subnetwork formed by centers {1, 2, …, M-1}

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Global indices

This time (t_r) can be described as the time that passes between the instant in which the job *exits* from M, to the instant in which it re-presents itself at the entrance of M

On this basis we define the ***cycle time*** $\boldsymbol{t_c}$ of the installation as the time that the job takes to make a complete turn, therefore

$$\boldsymbol{t_c = t_r + t_M}$$

where $\boldsymbol{t_M}$ is the response time of center M.

Global indices

On the basis of the definitions of t_r and t_c we may write expressions for their averages :

- $E(t_r(N))$, **mean response time** of the installation
- $E(t_c(N))$, **mean cycle time** of the installation

that we define **global indices** of the installation,

together with the following **index**:

- $\lambda_M(N)$, **mean throughput** of the installation

below introduced:

Global indices

We define and indicate with ***λ(N),*** the ***throughput*** *of the installation* ***,*** "**seen" by** center M, as the mean number of jobs per unit of time that exits the subnetwork {1, 2, …, M-1} and returns to M.

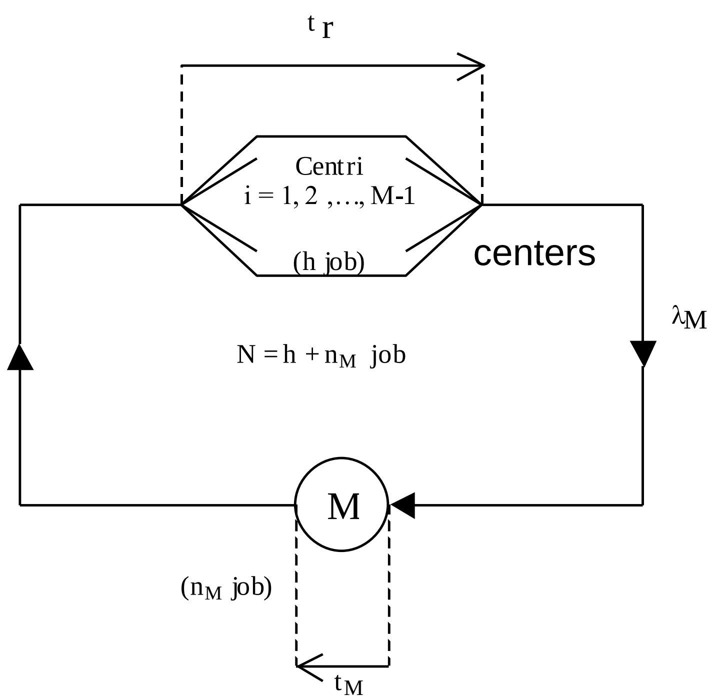
This index is the most simple to calculate in that, as we can see from the QN model illustration above, it coincides with ***throughput λ_M(N)*** of center M, obtainable as seen before for the **local** indices.

Global indices

For the calculation of the other two *global performance indices*, we consider that while the number of users in the network is fixed (N), in that we have a closed network, the same can't be said for the number $h = n_1 + n_2 + \ldots + n_{M-1}$ of jobs that, at a given time, populate the subnetwork {1, 2, …, M-1} and for the remaining n_M present in the center M.

Global indices

In order to calculate the global indices, we denote by *h* the population in the subnetwork {1, 2, …, M-1} at a given time



Global indices

Variable *h* behaves like a *random* variable, where we can write the mean

$$E(h(N)) = E(n_1(N)) + E(n_2(N)) + \ldots + E(n_{M-1}(N))$$

By applying Little's law, we obtain

$$E(t_r(N)) = \frac{E(h(N))}{\lambda_M(N)} = \frac{\sum_{i=1}^{M-1} E(n_i(N))}{\lambda_M(N)} = \sum_{i=1}^{M-1} \frac{\lambda_i(N)}{\lambda_M(N)} \; E(t_i(N))$$

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Global indices

Where the ratio $\lambda_i(N)/\lambda_M(N)$, denoted as ν_i, is called ***visits ratio*** and represents the **mean** number of times center *i* is visited by a job for each visit to the subnetwork (i.e. during its stay in the subnetwork)

Global indices

The ratios $\nu_i = \lambda_i(N)/\lambda_M(N)$ are obtained by the solution of the equations system

$$y_i(N) = \sum_{j=1}^{M} y_j(N)\ p_{ji} \qquad (i, j = 1, 2, \dots, M)$$

In vector form

$$\mathbf{y}(N) = \mathbf{y}(N)\ P$$

Global indices

where ***P*** is the MxM routing matrix $[p_{ij}]$ between the M centers of the QN, and y_i is the "relative" throughput of center ***i*** (i.e throughput relative to λ_M and to be to distinguished from the absolute throughput λ_i)

The solution of the system provides the ratios $\boldsymbol{y_i/y_j}$ and thus also the ratios $\boldsymbol{\lambda_i/\lambda_j}$ between the absolute throughputs.

The time complexity of the solution is $O(M^3)$

Global indices

By using the visits ratio v_i concept, the expression for the mean response time given above, can be rewritten:

$$E(t_r(N)) = \sum_{i=1}^{M-1} v_i \; E(t_i(N))$$

and called ***general response time theorem,*** which states that the **global** mean response time "seen" by center M is the weighed sum (according to the visits ratios) of the **local** mean response times of the (M-1) centers of the subnetwork.

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Global indices

The writing of the ***mean cycle time*** is now immediate:

$$E(t_c(N)) = E(t_r(N)) + E(t_M(N))$$

In other words, the sum of the mean global response time seen by center M and the mean local response time of center M itself.

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Global indices

And from this, by applying *Little's theorem* to all centers i=1, 2, …, M of the installation, we may obtain another formula (in addition to the $\lambda_M(N)$ given above) for the ***throughput*** *of the installation:*

$$\lambda(N) = \frac{N}{E(t_c(N))}$$

Global indices

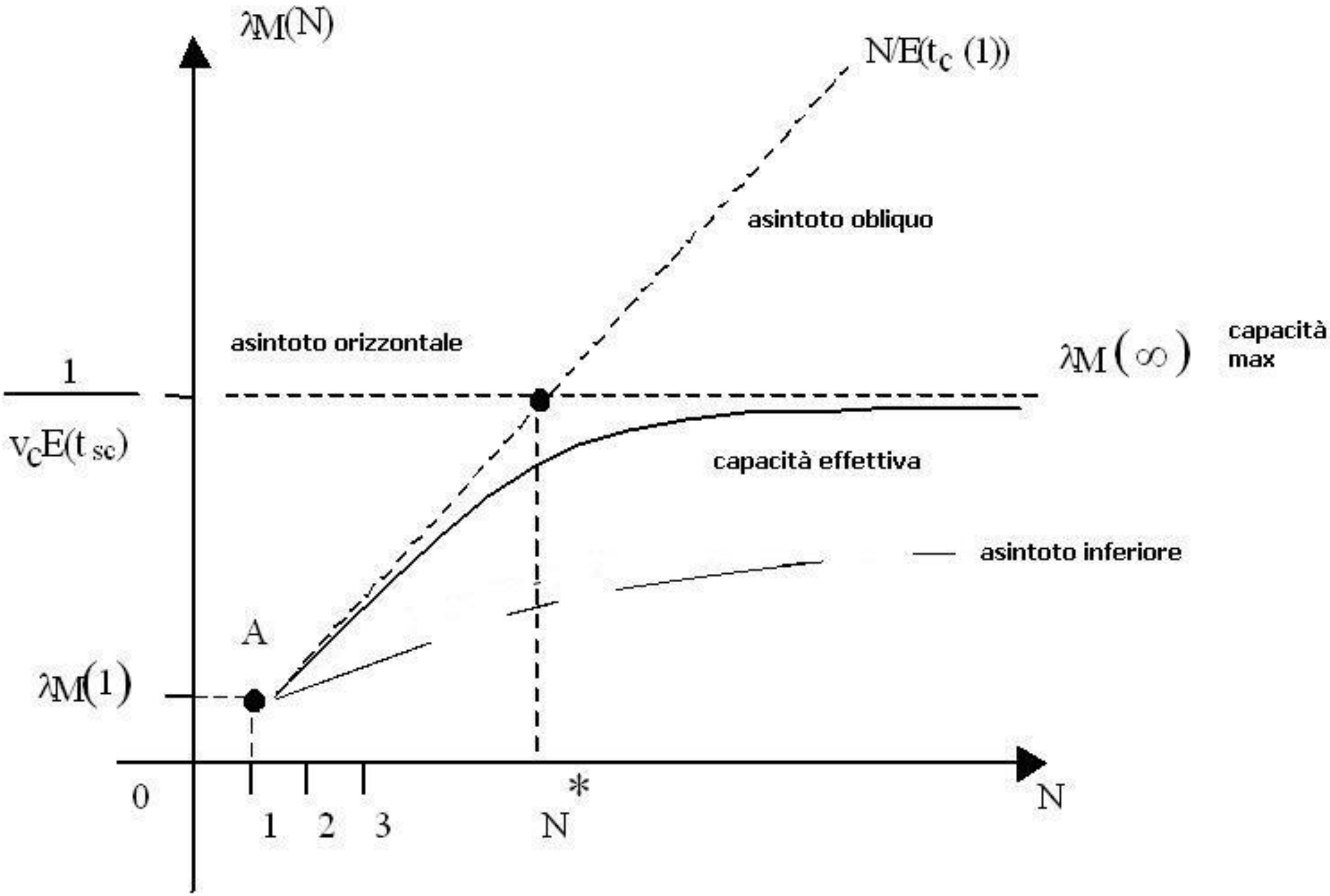
And, being $\lambda(N) = \lambda_M(N)$ we can also write $\boldsymbol{\lambda_M(N)}$, the *throughput* of the installation "seen" by center M, in the form:

$$\lambda_M(N) = \frac{N}{E(t_c(N))}$$

.Such a quantity shows an asymptotic curvilinear trend in function of N as in next illustration and discussed further on.

It is opportune to observe that the calculation of the denominator requires the entire solution of the model with either the Markovian method, or with other methods (e.g. product form).

Asymptotes of throughput



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